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Higher Order Three Derivative Runge-Kutta Method with Phase-Fitting and Amplification-Fitting Technique for Periodic IVPs

Ahmad, N. A. *[1], Senu, N.[1,2], Ibrahim, Z. B.[1,2], Othman, M.[1,3], and Ismail, Z.[1]

[1] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*
[2] *Department of Mathematics, Universiti Putra Malaysia, Malaysia*
[3] *Department of Communication Technology and Network, Universiti Putra Malaysia, Malaysia*

E-mail: nuramirah_ahmad@yahoo.com
* *Corresponding author*

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ABSTRACT

Two phase-fitted and amplification-fitted three derivative Runge-Kutta method (PFAFThDRK) for the numerical solution of first order initial value problems (IVPs) of higher algebraic order with oscillatory solutions are constructed. Using the phase-fitted and amplification-fitted property, a sixth-order three stage and seventh-order three stage three derivative Runge-Kutta (ThDRK) method are proposed. Using the same property of some existing Runge-Kutta methods (RK), the accuracy and efficiency of the methods constructed are compared by the means of numerical investigations.

Keywords: Diagonally Implicit methods, IVPs, ODEs, ThDRK methods, Phase-Fitted and Amplification-Fitted.

1. Introduction

The first order ordinary differential equations (ODEs) for the numerical solution of the IVPs are considered

$$y^{'} = f(x, y), \qquad\qquad y(x_0) = y_0, \tag{1}$$

where their solutions show periodically or oscillatory behavior. These type of problems appears throughout certain fields of applied sciences, for instance, mechanics, electronics, circuit simulation, orbital mechanics, astrophysics, molecular dynamics and etc. In general, periodically or oscillatory behavior problems are mostly known with second or higher order. Thence, it is essential to perform order reduction to solve the ODEs (1) by reducing them to first order problems.

Quantum cryptography is an extensive and rapidly growing field today in which quantum memory is implemented into quantum cryptography as a security mechanism within quantum key distribution (QKD). One essential fact is that implementing Schrödinger equations to quantum memory is crucial for further conceptual understanding as well as ensuring quantum security within quantum cryptography. The numerical study of Schrödinger equations based on ODEs system with periodic problem nowadays has drawn the attention of many applied mathematicians across the globe.

In fact, Anastassi and Simos (2012), Kosti et al. (2012) and Chen et al. (2012) efficiently solved the Schrödinger equation and related periodically problems by designing a new explicit phase-fitted and amplification-fitted for the optimization of the method. Nevertheless, Wu (2003) and Wu (2004) revealed that Schrödinger equations with spatial variables shows a crucial role in formulating potential scattering resulting in a significant decoherence in quantum memory. Those are basic tools in any study of quantum memory within quantum field scattering theory.

RK methods for solving oscillatory problems using several techniques, for instance, phase-fitted and amplification-fitted, trigonometrically-fitted and exponentially-fitted techniques have been developed and expanded by several famous authors such as Psihoyios and Simos (2005) and Anastassi and Simos (2005) in their written papers. Psihoyios and Simos (2005) developed a Runge-Kutta method with trigonometrically-fitting properties for the radial

Schrödinger equation of order five. Anastassi and Simos (2005) constructed two RK methods with trigonometrically-fitted property based on a classical England's RK method specially designed for radial Schrödinger equation of order five which have energy with lower powers in the local truncation error.

Recently, Fawzi et al. (2015) and Adel et al. (2016) derived two methods of phase-fitted and amplification-fitted for modified RK and RK method of order four respectively. Explicit two derivative Runge-Kutta (TDRK) methods given by Chan and Tsai (2010) in which they include the second derivative in its general formula making it "special". Only one evaluation of f is involved and a several number of g to be evaluated at every step. With this finding, they managed to derived methods up to order seven with five stages as well as some embedded pairs.

A TDRK method with trigonometrically-fitted for the numerical integration of radial Schrödinger equation and periodically problems of order five are constructed by Zhang et al. (2013). Other than that, Fang et al. (2013) and Chen et al. (2015) constructed two TDRK methods of order four and three practical TDRK methods with exponentially-fitted respectively. The newly derived methods are compared with some widely-known optimized codes as well as conventional exponentially-fitted RK methods in the literature.

Thus, in this research paper, a sixth-order and a seventh-order with both having three stages explicit three derivative Runge-Kutta (ThDRK) methods with phase-fitted and amplification-fitted are derived. In second section, an outline of ThDRK method is given and discussed. Meanwhile, in third section, phase-fitting and amplification-fitting properties are considered. The PFAFThDRK methods are constructed in fourth section. The numerical results, discussion and conclusion are discussed in fifth, sixth and seventh section respectively.

2. Three Derivative Runge-Kutta Methods

The scalar ODEs (1) is considered where $f : \Re^N \to \Re^N$. The second derivative are already known as in TDRK method. Hence, for this case, we assume the third derivative to be known where

$$
\begin{aligned}
y^{''} &= g(y) := f^{'}(y)f(y), \qquad g : \Re^N \to \Re^N, \qquad (2)\\
y^{'''} &= \hat{g}(y) := f^{''}(y)(f(y), f(y)) + f^{'}(y)f^{'}(y)f(y), \qquad \hat{g} : \Re^N \to \Re^N.
\end{aligned}
$$

For the numerical integration of IVPs (1), an explicit ThDRK method is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + h^2 \sum_{j=1}^{s} \hat{a}_{ij} g(Y_j) + h^3 \sum_{j=1}^{s} \hat{\hat{a}}_{ij} \hat{g}(Y_j), \tag{3}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i) + h^2 \sum_{i=1}^{s} \hat{b}_i g(Y_i) + h^3 \sum_{i=1}^{s} \hat{\hat{b}}_i \hat{g}(Y_i), \tag{4}$$

where $i = 1, \ldots, s$. Consider a minimum number of function evaluations for an explicit method,

$$Y_i = y_n + hc_i f(y_n) + \frac{1}{2}h^2 {c_i}^2 g(y_n) + h^3 \sum_{j=1}^{i-1} \hat{\hat{a}}_{ij} \hat{g}(Y_j), \tag{5}$$

$$y_{n+1} = y_n + hf(y_n) + \frac{1}{2}h^2 g(y_n) + h^3 \sum_{i=1}^{s} \hat{\hat{b}}_i \hat{g}(Y_i), \tag{6}$$

where $i = 2, \ldots, s$.

The explicit ThDRK method are shown below in the form of Butcher's Tableau.

$$\begin{array}{c||c} c & \hat{\hat{A}} \\ \hline & \hat{\hat{b}}^T \end{array} = \begin{array}{c||cccc} 0 & & & & \\ c_2 & \hat{\hat{a}}_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_s & \hat{\hat{a}}_{s1} & \cdots & \hat{\hat{a}}_{ss-1} & \\ \hline & \hat{\hat{b}}_1 & \cdots & \hat{\hat{b}}_{s-1} & \hat{\hat{b}}_s \end{array} \tag{7}$$

The unique and interesting part in this method is where only single evaluation of f is involved and a several number of g to be evaluated at every step in comparison to more evaluation of f and g at every step in conventional explicit RK methods and TDRK methods respectively. The stages number of the method is denoted as s and $\hat{\hat{a}}_{ij}, \hat{\hat{b}}_i$ and c_i are the ThDRK parameters which are assumed to be real. The s-dimensional vectors $\hat{\hat{b}}, c$ and $s \times s$ matrix, $\hat{\hat{A}}$ are introduced where $\hat{\hat{b}} = \left[\hat{\hat{b}}_1, \hat{\hat{b}}_2, \ldots, \hat{\hat{b}}_s\right]^T, c = [c_1, c_2, \ldots, c_s]^T$ and $\hat{\hat{A}} = \left[\hat{\hat{a}}_{ij}\right]$. We use the following simplifying assumption,

$$\sum_{i=1}^{s} \hat{\hat{a}}_{ij} = \frac{1}{6}{c_i}^3, \tag{8}$$

for $j = 1, \ldots, s$.

The order conditions and counts for ThDRK method are given in the following Table 1.

Table 1: Order conditions and counts for ThDRK methods

s	Order	Conditions		Unknowns	Conditions
1	3	$\hat{\hat{b}}^T e = \frac{1}{6}$		1	1
2	4	$\hat{\hat{b}}^T c = \frac{1}{24}$		3	2
2	5	$\hat{\hat{b}}^T c^2 = \frac{1}{60}$		3	3
3	6	$\hat{\hat{b}}^T c^3 = \frac{1}{120}$		6	4
3	7	$\hat{\hat{b}}^T c^4 = \frac{1}{210}$,	$\hat{\hat{b}}^T \hat{\hat{A}} c = \frac{1}{5040}$	6	6

3. Phase-Fitted and Amplification-Fitted Three Derivative Runge-Kutta Method

The linear scalar equation below is considered,

$$y^{'} = \lambda y. \tag{9}$$

The exact solution with initial value $y(x_0) = y_0$ of this equation satisfies

$$y(x_0 + h) = H_0(z) y_0, \tag{10}$$

where $H_0(z) = \exp(z)$ and $z = iv$. A phase advance $v = \lambda h$ is experienced by the exact solution and the amplification remains steady after a cycle of time h. The ThDRK method is applied to the test equation (9) to yield

$$y_1 = H(z) y_0, \tag{11}$$

where

$$\begin{aligned} H(z) =& 1 + v + \frac{1}{2}v^2 + v^3 \hat{b} \left(I - v^2 \hat{\hat{A}}\right)^{-1} e + v^4 \left(I - v^2 \hat{\hat{A}}\right)^{-1} c + \\ & \frac{1}{2} v^5 \left(I - v^2 \hat{\hat{A}}\right)^{-1} c^2, \end{aligned} \tag{12}$$

where $e = [1, \ldots, 1]^T, c = [c_1, \ldots, c_s]^T$ and $c^2 = \left[c_1^2, \ldots, c_s^2\right]^T$.

The stability polynomial of the ThDRK method is given by $H(z)$ as mentioned in Turacı and Öziş (2017). $U(v)$ and $V(v)$ denote the real and imaginary part of $H(z)$ respectively. For small h, $argH(z) = tan^{-1}\left(\frac{V(v)}{U(v)}\right)$ and $|H(z)| = \sqrt{U^2(v) + V^2(v)}$. According to the analysis above, the following definition arises.

Definition 3.1. *(van der Houwen and Sommeijer (1987))*
The quantities

$$\tilde{P}(v) = v - argH(z) \qquad \text{and} \qquad \tilde{D}(v) = 1 - |H(z)|, \tag{13}$$

are called the phase lag (or dispersion) and the error of amplification factor (or dissipation) of the method, respectively. If

$$\tilde{P}(v) = c_\phi v^{\alpha+2} + \mathcal{O}(v^{\alpha+4}) \qquad \text{and} \qquad \tilde{D}(v) = c_d v^{\beta+2} + \mathcal{O}(v^{\beta+4}), \tag{14}$$

then the method is called dispersive of order α and dissipative of order β, respectively. If

$$\tilde{P}(v) = 0 \qquad \text{and} \qquad \tilde{D}(v) = 0, \tag{15}$$

the method is called phase-fitted (or zero-dispersive) and amplification-fitted (or zero dissipative), respectively.

Theorem 3.1. *(Chen et al. (2012))*
The method is phase-fitted and amplification-fitted if and only if

$$U(v) = \cos(v) \qquad \text{and} \qquad V(v) = \sin(v). \tag{16}$$

4. Derivation of the Methods with Phase-Fitting and Amplification-Fitting Properties

Two PFAFThDRK methods, which are sixth-order three stages and seventh-order three stages ThDRK methods are constructed. The derivation of these methods are discussed in the following subsection.

4.1 Sixth-order Three Stages PFAFThDRK Method

For sixth-order ThDRK method, the steps to derive the method is very simple and easy. The order conditions are solved for up to order six as given in Table 1. A ThDRK formula of sixth-order method carry four equations and six unknowns. Thus, this system has two free parameters. Evaluating the simplifying assumption (8) leads to

$$\hat{\hat{a}}_{21} = \frac{c_2^3}{6}, \tag{17}$$

$$\hat{\hat{a}}_{31} = \frac{c_3^3}{6} - \hat{\hat{a}}_{32}, \tag{18}$$

According to order conditions up to order six in Table 1, we have

$$\hat{\hat{b}}_1 + \hat{\hat{b}}_2 + \hat{\hat{b}}_3 - \frac{1}{6} = 0, \tag{19}$$

$$\hat{\hat{b}}_2 c_2 + \hat{\hat{b}}_3 c_3 - \frac{1}{24} = 0, \tag{20}$$

$$\hat{\hat{b}}_2 {c_2}^2 + \hat{\hat{b}}_3 {c_3}^2 - \frac{1}{60} = 0, \tag{21}$$

$$\hat{\hat{b}}_2 {c_2}^3 + \hat{\hat{b}}_3 {c_3}^3 - \frac{1}{120} = 0. \tag{22}$$

Solve (19)–(22) will lead to a solution of $\hat{\hat{b}}_1, \hat{\hat{b}}_2, \hat{\hat{b}}_3$ and c_3 in term of c_2

$$\hat{\hat{b}}_1 = \frac{1}{120}\,\frac{15\,{c_2}^2 - 10\,c_2 + 1}{c_2\,(2\,c_2 - 1)}, \tag{23}$$

$$\hat{\hat{b}}_2 = \frac{1}{120}\,\frac{1}{c_2\,(-4\,c_2 + 1 + 5\,{c_2}^2)}, \tag{24}$$

$$\hat{\hat{b}}_3 = \frac{1}{120}\,\frac{(5\,c_2 - 2)\left(4 - 20\,c_2 + 25\,{c_2}^2\right)}{(2\,c_2 - 1)\,(-4\,c_2 + 1 + 5\,{c_2}^2)}, \tag{25}$$

$$c_3 = \frac{2\,c_2 - 1}{5\,c_2 - 2}. \tag{26}$$

For simplicity, $\hat{a}_{32} = 0$ is chosen. The principal local truncation error coefficient, $\left\|\tau^{(7)}\right\|_2$ must have a very small value and this can be done by choosing an appropriate value of c_2. A vast global error difference will result in the wrong choices of c_2. By plotting the graph of $\left\|\tau^{(7)}\right\|_2$ against c_2, a small value of c_2 is chosen in the range of $[0.0, 1.0]$ and hence, the value of c_2 lies between $[0.0, 1.0]$. The value of $c_2 = \frac{1}{100}$ is chosen for an optimized pair. The method derived with its coefficients are showed below in Table 2. It is denoted as ThDRK(3,6).

Table 2: Sixth-order ThDRK method

0	0		
$\frac{1}{100}$	$\frac{1}{6000000}$	0	
$\frac{98}{195}$	$\frac{470596}{22244625}$	0	0
	$-\frac{601}{784}$	$\frac{5000}{5763}$	$\frac{98865}{1506064}$

The maximum global error is optimized with the combination of free parameters, $\hat{\hat{b}}_1$ and $\hat{\hat{b}}_2$. In Table 2, all the coefficients are used to evaluate the stability polynomial $H(z)$. Thenceforth, the separation of $H(z)$ into real and imaginary part leads to,

$$\cos(v) = -\frac{1}{6000000}\,\hat{\hat{b}}_2 v^6 - \frac{2401}{1728900}\,v^6 + \frac{1}{100}\,\hat{\hat{b}}_2 v^4 + \frac{507}{15368}\,v^4 - \frac{1}{2}\,v^2 + 1, \tag{27}$$

$$\sin(v) = \frac{1}{20000}\,\hat{\hat{b}}_2 v^5 + \frac{637}{76840}\,v^5 - \hat{\hat{b}}_1 v^3 - \frac{98865}{1506064}\,v^3 - \hat{\hat{b}}_2 v^3 + v. \tag{28}$$

Solving equation (27) and (28) we will obtain

$$\begin{aligned}\hat{\hat{b}}_1 = &-\frac{1}{11760\,(-60000+v^2)\,v^4}\Big(-92256025\,v^6 + 4802\,v^8 - 705600000\,\sin(v)\,v + \\ &11760\,\sin(v)\,v^3 + 2283252240\,v^4 - 34577928000\,v^2 + 70560000000 + \\ &3528000\,v^2\cos(v) - 70560000000\,\cos(v)\Big),\end{aligned} \tag{29}$$

$$\hat{\hat{b}}_2 = -\frac{10000}{5763}\left(\frac{-3457800 + 1728900\,v^2 + 4802\,v^6 - 114075\,v^4 + 3457800\,\cos(v)}{v^4\,(-60000+v^2)}\right). \tag{30}$$

The following Taylor expansions as $v \to 0$ are obtained:

$$\begin{aligned}\hat{\hat{b}}_1 =& -\frac{601}{784} - \frac{23}{10080}\,v^4 + \frac{43}{1728000}\,v^6 - \frac{133999}{725760000000}\,v^8 + \\ &\frac{3936606091}{3962649600000000000}\,v^{10} + \dots,\\ \hat{\hat{b}}_2 =& \frac{5000}{5763} + \frac{5}{2016}\,v^4 - \frac{1997}{72576000}\,v^6 + \frac{3326011}{15966720000000}\,v^8 - \\ &\frac{3021131303}{2641766400000000000}\,v^{10} + \dots.\end{aligned}$$

This derived method is called as PFAFThDRK(3,6). PFAFThDRK(3,6) will reduce back to its original method as $v \to 0$. Otherwise, PFAFThDRK(3,6) will have the same error constant as ThDRK(3,6) whenever $v \to 0$.

4.2 Seventh-order Three Stages PFAFThDRK Method

The coefficient of the existing ThDRK method given in Table 3 is considered.

Table 3: Butcher Tableau for ThDRK(3,7) method

$$
\begin{array}{c||ccc}
0 & 0 & & \\
\frac{3}{7}-\frac{\sqrt{2}}{7} & -\frac{\left(-3+\sqrt{2}\right)^3}{2058} & 0 & \\
\frac{3}{7}+\frac{\sqrt{2}}{7} & \frac{71}{14406}+\frac{61\sqrt{2}}{14406} & \frac{122}{7203}+\frac{71\sqrt{2}}{7203} & 0 \\
\hline
 & \frac{1}{30} & \frac{1}{15}+\frac{13\sqrt{2}}{480} & \frac{1}{15}-\frac{13\sqrt{2}}{480}
\end{array}
$$

The maximum global error is optimized with the combination of free parameters, $\hat{\hat{b}}_1$ and $\hat{\hat{b}}_2$. In Table 3, all the coefficients are used to evaluate the stability polynomial $H(z)$. Thenceforth, the separation of $H(z)$ into real and imaginary part leads to,

$$
\begin{aligned}
\cos(v) =& 1-\frac{1}{2}v^2+\left(-\frac{1}{7}\hat{\hat{b}}_2\sqrt{2}-\frac{\sqrt{2}}{480}+\frac{3}{7}\hat{\hat{b}}_2+\frac{1}{48}\right)v^4+\left(\frac{29\,\hat{\hat{b}}_2\sqrt{2}}{2058}-\frac{\sqrt{2}}{2880}-\right.\\
&\left.\frac{15\,\hat{\hat{b}}_2}{686}-\frac{1}{1440}\right)v^6+\left(-\frac{\sqrt{2}}{70560}+\frac{1}{23520}\right)v^8,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\sin(v) =& v+\left(-\hat{\hat{b}}_2-\hat{\hat{b}}_1+\frac{13\sqrt{2}}{480}-\frac{1}{15}\right)v^3+\left(-\frac{3\,\hat{\hat{b}}_2\sqrt{2}}{49}+\frac{\sqrt{2}}{960}+\frac{11\,\hat{\hat{b}}_2}{98}+\frac{1}{240}\right)v^5\\
&-\frac{v^7}{5040}+\left(\frac{11}{1481760}-\frac{\sqrt{2}}{246960}\right)v^9.
\end{aligned} \tag{32}
$$

Solving equation (31) and (32), the following Taylor expansions as $v \to 0$ are obtained

$$
\begin{aligned}
\hat{\hat{b}}_1 =& \frac{1}{30}+\frac{\left(4\sqrt{2}-5\right)v^4}{-120960+40320\sqrt{2}}-\frac{\left(-727+499\sqrt{2}\right)v^6}{25401600\left(-3+\sqrt{2}\right)^2}-\\
&\frac{\left(19656918\sqrt{2}-27814993\right)v^8}{1150082841600\left(-3+\sqrt{2}\right)^3}+\frac{\left(-32049357357+22660009499\sqrt{2}\right)v^{10}}{4395616620595200\left(-3+\sqrt{2}\right)^4}+\ldots,
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\hat{\hat{b}}_2 =& \frac{1}{15}+\frac{13\sqrt{2}}{480}-\frac{\left(4\sqrt{2}-5\right)v^4}{-120960+40320\sqrt{2}}+\frac{\left(-3942+2609\sqrt{2}\right)v^6}{88905600\left(-3+\sqrt{2}\right)^2}-\\
&\frac{\left(-14278451+10094970\sqrt{2}\right)v^8}{1150082841600\left(-3+\sqrt{2}\right)^3}+\frac{\left(1129876894\sqrt{2}-1598038383\right)v^{10}}{439561662059520\left(-3+\sqrt{2}\right)^4}+\ldots.
\end{aligned} \tag{34}
$$

This derived method is called as PFAFThDRK(3,7). PFAFThDRK(3,7) will reduce back to its original method as $v \to 0$. Otherwise, PFAFThDRK(3,7) will have the same error constant as ThDRK(3,7) whenever $v \to 0$.

5. Problems Tested and Numerical Results

The derived methods PFAFThDRK(3,6) and PFAFThDRK(3,7) are compared in term of their numerical performances with some famous existing RK and TDRK methods by considering Problems 1–4 below. C codes are used for solving differential equations where all the problems choosen are having oscillatory solutions.

Problem 1 (Two-body problem, Simos (2005))

$$
\begin{aligned}
{y_1}' &= y_2, && y_1(0) = 1,\\
{y_2}' &= -\frac{y_1}{(\sqrt{{y_1}^2 + {y_3}^2})^3}, && y_2(0) = 0,\\
{y_3}' &= y_4, && y_3(0) = 0,\\
{y_4}' &= -\frac{y_3}{(\sqrt{{y_1}^2 + {y_3}^2})^3}, && y_4(0) = 1.
\end{aligned}
$$

Exact solution is

$$y_1(x) = \cos(x), \quad y_2(x) = -\sin(x), \quad y_3(x, = \sin(x), \quad y_4(x) = \cos(x).$$

Problem 2 (Prothero-Robinson problem, Chan and Tsai (2010))

$$y' = -\lambda(y - \varphi) + \varphi', \qquad y(0) = \varphi(0), \qquad Re(\lambda) < 0,$$

where $\varphi(x)$ is a smooth function and $\varphi(x) = \sin(x)$.
Exact solution is $y(x) = \varphi(x)$.

Problem 3 (Inhomogeneous problem, Van de Vyver (2006))

$$
\begin{aligned}
{y_1}' &= y_2, && y_1(0) = 1,\\
{y_2}' &= -100y_1 + 99\sin(x), && y_2(0) = 11.
\end{aligned}
$$

Exact solution is

$$y_1(x) = \cos(10x) + \sin(10x) + \sin(x), \quad y_2(x) = -10\sin(x) + 10\cos(10x) + \cos(x).$$

Problem 4 (An "almost" Periodic Orbit problem, Stiefel and Bettis (1969))

$$
\begin{aligned}
{y_1}' &= y_2, & y_1(0), &= 1,\\
{y_2}' &= -y_1 + 0.001\cos(x), & y_2(0) &= 1,\\
{y_3}' &= y_4, & y_3(0) &= 0,\\
{y_4}' &= -y_3 + 0.001\sin(x), & y_4(0) &= 0.995.
\end{aligned}
$$

Exact solution is

$$
\begin{aligned}
y_1(x) &= \cos(x) + 0.0005x\sin(x), \quad y_2(x) = -\sin(x) + 0.0005x\cos(x) + 0.0005\sin(x),\\
y_3(x) &= \sin(x) - 0.0005x\cos(x), \quad y_4(x) = \cos(x) + 0.0005x\sin(x) - 0.0005\cos(x).
\end{aligned}
$$

Figures 1–4 used the following notations:

- **PFAFThDRK(3,6):** The sixth-order three stages ThDRK method with phase-fitting and amplification-fitting properties derived earlier.
- **PFAFThDRK(3,7):** The seventh-order three stages ThDRK method with phase-fitting and amplification-fitting properties derived earlier.
- **TFTDRK(3,5):** Existing three stage order five trigononometrically-fitted TDRK method derived by Zhang et al. (2013).
- **TFRKS(6,5):** Existing six stages order five trigononometrically-fitted RK method developed by Anastassi and Simos (2005).
- **TFRKAS(6,5):** Existing six stages order five trigononometrically-fitted RK method constructed by Anastassi and Simos (2005).
- **PFAFRKC(7,5):** Existing seven stages order five phase-fitted and amplification-fitted RK method given in Chen et al. (2012).

The numerical performances are represented graphically in Figures 1–4.

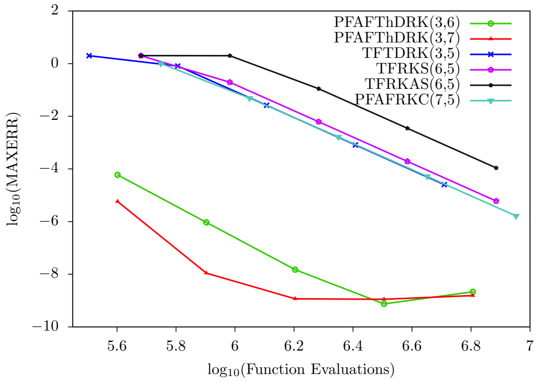

Figure 1: The performance curve for the two-body problem (Problem 1) for PFAFThDRK(3,6) and PFAFThDRK(3,7) methods with $T = 10000$ and time step $h = 1/2^i, i = 3, \ldots, 7$.

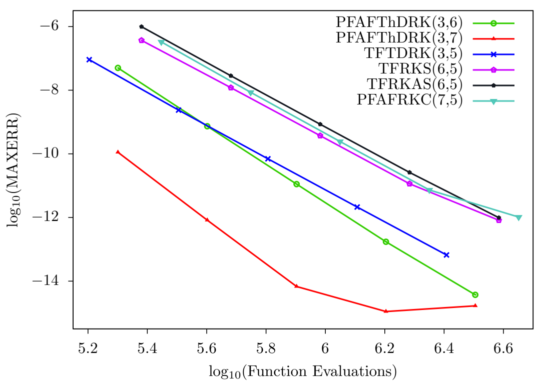

Figure 2: The performance curve for Prothero-Robinson problem (Problem 2) for PFAFThDRK(3,6) and PFAFThDRK(3,7) methods with $\lambda = 1, T = 10000$ and time step $h = 1/2^i, i = 2, \ldots, 6$.

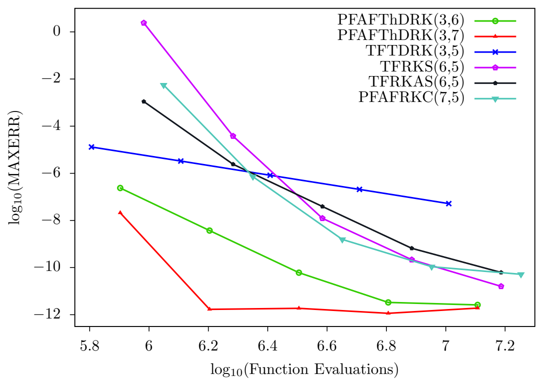

Figure 3: The performance curve for the inhomogeneous problem (Problem 3) for PFAFThDRK(3,6) and PFAFThDRK(3,7) methods with $T = 10000$ and time step $h = 1/2^i, i = 4, \dots, 8$.

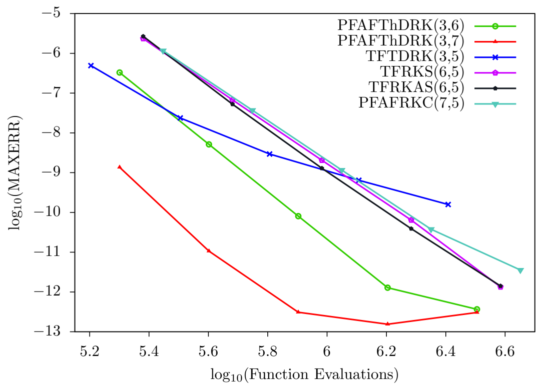

Figure 4: The performance curve for "almost" periodic problem (Problem 4) for PFAFThDRK(3,6) and PFAFThDRK(3,7) methods with $T = 10000$ and time step $h = 1/2^i, i = 2, \dots, 6$.

6. Discussion

The numerical experiments show the properties of the phase-fitted and amplification-fitted ThDRK methods, PFAFThDRK(3,6) and PFAFThDRK(3,7) derived earlier in this paper. The comparison are made between the proposed methods and some existing RK and TDRK methods with trigonometrically-fitting, phase-fitting and amplification-fitting properties. The graphs are plotted between the global maximum error and the efficiency of the proposed method over a higher integration period. Figures 1–4 represent the accuracy and efficiency of the proposed method developed by plotting the graph of the maximum global error against the number of function evaluations, both in logarithm for a lengthy periods of computations. From the plotted graphs, PFAFThDRK(3,6) and PFAFThDRK(3,7) show higher accuracy with the smallest maximum global error in comparison with the same type of other existing RK and TDRK methods.

In Figure 3, the smaller the h value, the maximum global error of the PFAFThDRK(3,6) and PFAFThDRK(3,7) methods seem to flatten throughout the curve end. The frequency, λ and step-size, h determine the accuracy of the method. The derived methods will converge to its original method as the value of h becomes smaller. The comparisons are made with lower order methods since there is no RK methods with such properties which have order higher than five. There are plenty of RK methods with order higher than five but they did not have the same properties as the derived methods but this kind of comparison will become unfair and irrelevant. The comparisons are made between methods of the same properties but it can be seen that PFAFThDRK(3,6) and PFAFThDRK(3,7) methods are the most accurate methods of all in term of maximum global error. PFAFThDRK(3,6) and PFAFThDRK(3,7) methods have lesser number of stages compared to other existing RK methods.

7. Conclusion

In this research, sixth and seventh-order PFAFDIThDRK methods have been developed. To sum up, the constructed methods are more promising compared to other existing well-known RK and TDRK methods with trigonometrically-fitting, phase-fitting and amplification-fitting properties in the literature in terms of accuracy, efficiency and the number of function evaluations at every step based on the numerical results obtained.

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